2016/17 MATH2230B/C Complex Variables with Applications
Suggested Solution of Selected Problems in HW 4
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P.247 4 and P.264 9 will be graded

All the problems are from the textbook, Complex Variables and Application ($9th$ edition).

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1. Find the residue at $z = 0$ of the function

   (a) $\dfrac{1}{z+z^2}$;

   (b) $z\cos\left(\dfrac{1}{z}\right)$;

   (c) $\dfrac{z-\sin z}{z}$;

   (d) $\dfrac{\cot z}{z^4}$;

   (e) $\dfrac{\text{sinh}z}{z^4(1-z^2)}$.

   Solution. *One can find the Taylor or Laurent expansion of a given function to calculate the residue at some certain point $z = z_0 \in \mathbb{C}$.*

   (a) Note that

$$\frac{1}{z+z^2} = \frac{1}{z}\cdot\frac{1}{1+z} = \frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n z^{n-1},$$

   then the residue at $z = 0$ is 1.

   (b) Note that

$$z\cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}\frac{1}{z^{2n-1}} = z - \frac{1}{2z} + \cdots ,$$

   thus the residue at $z = 0$ is $-\frac{1}{2}$.

   (c) Note that

$$\frac{z-\sin z}{z} = 1 - \frac{\sin z}{z}$$

   is analytic at $z = 0$, then the residue is 0.

   (d) Note that

$$\frac{\cot z}{z^4} = \frac{1}{z^5} - \frac{1}{3z^3} - \frac{1}{45z} + \cdots ,$$

   thus the residue at $z = 0$ is $-\frac{1}{45}$. One can use the definition of $\cot z$

$$\cot z = \frac{\cos z}{\sin z} = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots$$

to find the coefficients of the series if the expansions of $\sin z$ *and* $\cos z$ *are known.*

(e) *Note that*

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^3} + \frac{7}{6z} + \cdots ,$$

thus the residue at $z = 0$ *is* $\frac{7}{6}$*. One can use the definition of sinhz*

$$\sinh z = \frac{e^z - e^{-z}}{2} = z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots$$

to find the expansion and we know that

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}.$$

2. Use Cauchy's residue theorem (Sec. 76) to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

(a) $\dfrac{\exp(-z)}{z^2}$;

(b) $\dfrac{\exp(-z)}{(z-1)^2}$;

(c) $z^2 \exp\left(\dfrac{1}{z}\right)$;

(d) $\dfrac{z+1}{z^2-2z}$.

Solution. *To calculate the integral around the circle* $|z| = 3$ *in the positive sense, the residue need to be calculated first.*

(a) *The singularity of the given function is* $z = 0$*, and the residue is* -1*, thus*

$$\int_C \frac{\exp(-z)}{z^2} dz = -2\pi i.$$

Note that at $z = 0$*,*

$$\frac{\exp(-z)}{z^2} = \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2} - \frac{z}{6} + \cdots .$$

(b) *The singularity of the given function is* $z = 1$*, and the residue is* $-\frac{1}{e}$*, thus*

$$\int_C \frac{\exp(-z)}{(z-1)^2} dz = \frac{-2\pi i}{e}.$$

Note that at $z = 1$*,*

$$\frac{\exp(-z)}{(z-1)^2} = \frac{1}{e(z-1)^2} - \frac{e}{(z-1)} + \frac{1}{2e} + \cdots .$$

(c) The singularity of the given function is $z = 0$, and the residue is $\frac{1}{6}$, thus

$$\int_C z^2 \exp\left(\frac{1}{z}\right) = \frac{2\pi i}{6} = \frac{\pi i}{3}.$$

Note that at $z = 0$,

$$z^2 \exp\left(\frac{1}{z}\right) = z^2 + z + \frac{1}{2} + \frac{1}{6z} + \frac{1}{24z^2} + \cdots .$$

(d) The singularties of the given function are $z = 0$ and $z = 2$, the residues are $-\frac{1}{2}$ and $\frac{3}{2}$ respectively, then

$$\int_C \frac{z+1}{z^2 - 2z} dz = \left(-\frac{1}{2} + \frac{3}{2}\right) 2\pi i = 2\pi i.$$

Note that at $z = 0$,

$$\frac{z+1}{z^2 - 2z} = -\frac{1}{2z} - \frac{3}{4} - \frac{3z}{8} + \cdots ,$$

and at $z = 2$,

$$\frac{z+1}{z^2 - 2z} = \frac{3}{2(z-2)} - \frac{1}{4} + \frac{z-2}{8} + \cdots .$$

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4. Use the theorem in Sec. 77, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

   (a) $\dfrac{z^5}{1 - z^3}$;

   (b) $\dfrac{1}{1 + z^2}$;

   (c) $\dfrac{1}{z}$.

   Solution.

(a) Let $f(z) = \dfrac{z^5}{1 - z^3}$. The residue of $z^{-2} f\left(\dfrac{1}{z}\right)$ at $z = 0$ is -1. Note that at $z = 0$,

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = -\frac{1}{z^4} - \frac{1}{z} - z^2 + \cdots .$$

Then the integral of the function around $|z| = 2$ is

$$\int_{|z|=2} f(z) dz = -2\pi i.$$

(b) Let $f(z) = \dfrac{1}{1 + z^2}$. Then $z^{-2} f\big(\dfrac{1}{z}\big) = f(z)$ and the function is analytic at $z = 0$, then the integral around $|z| = 2$ is zero.

(c) Let $f(z) = \frac{1}{z}$*. Then* $z^{-2} f\left(\frac{1}{z}\right) = \frac{1}{z}$*. Then the integral around* $|z| = 2$ *is*

$$\int_{|z|=2} f(z)dz = 2\pi i.$$

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1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a removable singular point, an essential singular point, or a pole:

   (a) $z \exp\left(\frac{1}{z}\right)$;

   (b) $\frac{z^2}{1+z}$;

   (c) $\frac{\sin z}{z}$;

   (d) $\frac{\cos z}{z}$;

   (e) $\frac{1}{(2-z)^3}$.

   Solution. *(a) The principal part of the function at its isolated point* $z = 0$ *is*

   $$\frac{1}{2z} + \frac{1}{6z^2} + \frac{1}{24z^3} + \cdots .$$

   Then, that point is an essential singular point.

   (b) The principal part of the function at its isolated point $z = -1$ *is*

   $$\frac{1}{z+1}.$$

   Then, it is a simple pole.

   (c) The principal part of the function at its isolated point $z = 0$ *is zero. Then, it is a removable singular point.*

   (d) The principal part of the function at its isolated point $z = 0$ *is*

   $$\frac{1}{z}.$$

   Then, it is a simple pole.

   (e) The principal part of the function at its isolated point $z = 2$ *is the function itself. Then, it is a pole of order* 3.

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(a) $\dfrac{1-\cosh z}{z^3}$;

(b) $\dfrac{1-\exp(2z)}{z^4}$;

(c) $\dfrac{\exp(2z)}{(z-1)^2}$.

Solution. *(a) The Laurent series representation of the function at $z=0$ is*

$$-\frac{1}{2z}-\frac{z}{24}-\frac{z^3}{720}-\cdots.$$

The singular point is then a simple pole. The residue $B=-1/2$.

(b) The principal part of the Laurent series representation of the function at $z=0$ is

$$-\frac{2}{z^3}-\frac{2}{z^2}-\frac{4}{3z}.$$

Then, the singular point is a pole of order 3. The residue $B=-4/3$.

(c) The principal part of the Laurent series representation of the function at $z=1$ is

$$\frac{e^2}{(z-1)^2}+\frac{2e^2}{(z-1)}.$$

Then, it is a pole of order 2. The residue $B=2e^2$.

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3. In each case, find the order m of the pole and the corresponding residue B at the singularity $z=0$:

(a) $\dfrac{\sinh z}{z^4}$;

(b) $\dfrac{1}{z(e^z-1)}$.

Solution. *(a) The principal part of the given function is (at $z=0$)*

$$\frac{1}{z^3}+\frac{1}{6z}.$$

The order is $m=3$ and the residue $B=\frac{1}{6}$.

(b) The principal part of the given function is (at $z=0$)

$$\frac{1}{z^2}-\frac{1}{2z}.$$

The order is $m=2$ and the residue $B=-\frac{1}{2}$.

4. Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,\mathrm{d}z,$$

taken counterclockwise around the circle (a) $|z-2|=2$; (b) $|z|=4$.

Solution. *(a) The singular point inside the circle is $z=1$ and it is a simple pole. The residue at $z=1$ is*

$$\frac{3z^3+2}{(z^2+9)} = \frac{3+2}{1+9} = \frac{1}{2}.$$

Therefore,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,\mathrm{d}z = \frac{2\pi i}{2} = \pi i.$$

(b) The singular points inside the circle are $z=1$, $z=3i$ and $z=-3i$, they are all simple poles. The residues are $1/2$ and

$$\frac{3z^3+2}{(z-1)(z+3i)} = \frac{3(3i)^3+2}{(3i-1)(6i)} = \frac{2-81i}{(3i-1)(6i)} \qquad z=3i$$

and

$$\frac{3z^3+2}{(z-1)(z-3i)} = \frac{3(-3i)^3+2}{(-3i-1)(-6i)} = \frac{2+81i}{(3i+1)(6i)} \qquad z=-3i.$$

Hence,

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,\mathrm{d}z = \left(\frac{1}{2}+\frac{2+81i}{(3i+1)(6i)}+\frac{2-81i}{(3i-1)(6i)}\right)2\pi i = \left(\frac{1}{2}+\frac{5}{2}\right)2\pi i = 6\pi i.$$

5. Find the value of the integral

$$\int_C \frac{\mathrm{d}z}{z^3(z+4)},$$

taken counterclockwise around the circle (a) $|z|=2$; (b) $|z+2|=3$.

Solution. *The residue of the integrand at $z=0$ is*

$$\frac{1}{2}\left(\frac{1}{z+4}\right)'' = \frac{1}{(z+4)^3} = \frac{1}{64}.$$

The residue of the integrand at $z=-4$ is

$$\frac{1}{z^3} = -\frac{1}{64}.$$

(a) The singular point $z=0$ is inside the circle, hence the integral is

$$\int_C \frac{dz}{z^3(z+4)} = -\frac{2\pi i}{64} = -\frac{\pi i}{32}.$$

(b) The singular points $z=0$ and $z=-4$ are inside the circle, hence the integral is

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i\left(\frac{1}{64}-\frac{1}{64}\right) = 0.$$